Charles University in Prague
Faculty of Mathematics and Physics

MASTER THESIS

Bc. Michaela Seifrtová

Regular Coverings
Structure and Complexity

Regulární nakrytí - struktura a složitost

Department of Applied Mathematics

Supervisor of the master thesis:	Doc. RNDr. Jiří Fiala, Ph.D.
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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Regulární nakrytí - struktura a složitost

Autor: Bc. Michaela Seifrtová

Katedra: Katedra aplikované matematiky

Vedoucí diplomové práce: Doc. RNDr. Jiří Fiala, Ph.D.
Katedra aplikované matematiky

Abstrakt: Diplomová práce se sestává ze dvou hlavních částí, první zaměřené na strukturu nakrytí grafů, ve které jsou prezentovány různé vlastnosti regulárních nakrytí, a druhé pojednávájící o výpočetní složitosti problému nakrytí grafů. V této oblasti byly dosaženy příznivé výsledky, zejména bylo dokázáno, že problém regulárního nakrytí je řešitelný v polynomiálním čase pro všechny grafy, jejichž řád je prvočíselným násobkem řádu nakrývaného grafu.

Klíčová slova: Regulární nakrytí, fundamentální grupa, grupa transformací nakrytí, přiřazení napětí, výpočetní složitost;

Title: Regular Coverings - Structure and Complexity

Author: Bc. Michaela Seifrtová

Department: Department of Applied Mathematics

Supervisor: Doc. RNDr. Jiří Fiala, Ph.D.
Department of Applied Mathematics

Abstract: The thesis consists of two main parts, the first concentrated on the structure of graph coverings, where different properties of regular graph coverings are presented, and the second dealing with computational complexity of the covering problem. Favorable results have been achieved in this area, proving the problem is solvable in polynomial time for all graphs whose order is a prime multiple of the order of the covered graph.

Keywords: Regular covering, fundamental group, covering transformation group, voltage assignment, computational complexity;

Contents

Introduction

The subject of this thesis are coverings. Although the term is quite perspicuous - having two structures, one tries to find a mapping (covering) from the bigger one onto the smaller one, covering it completely - there are many interesting issues related to this topic. There are many conditions one may give on the covering projection - injectivity, surjectivity, or the condition of being a homomorphism. Indispensable is also the question of the very existence of a covering projection between the two structures.

Our attention will be focused on coverings of graphs, which will be locally bijective homomorphisms. In other words, we will require them to keep local incidences and map the neighborhood of each vertex onto the neighborhood of its image. Special case of coverings are regular coverings, which will be of the greatest interest in this work.

The origin of the concept of coverings traces back to the beginning of the twentieth century to Reidemeister, and Grottendieg (1950). Later in 1970-ties, knowing, that there existed a graph of certain degree of symmetry, Conway and Biggs used the construction of covering graphs to find an infinite class of such graphs, reader can find it in [4]. Massey studied topological spaces and their coverings, and dedicated one chapter of his book [18] to graph coverings, which allowed them to be studied from a different perspective. As showed by Gross and Tucker [9], [10] there is also a profound connection between graph coverings and voltage spaces. Worthy of remark is their theorem, which says that every covering projection can be constructed by permutation voltage assignments. Later Hofmeister [11] classified the isomorphism classes of covering projections, and Biggs published the article [3], in which he used homology groups of graphs to construct covering graphs, which clarifies the classification problem of cubic graphs, and showed that some of the finitely presented groups are, in fact, infinite.

In this work we will try to follow all of the various approaches to the concept of graph covering, show their advantages and some of the main results achieved in this area. In the second chapter we will focus on regular coverings, show several definitions of this term and how they cohere.

Another interesting question related to coverings is the question of their computational complexity. Probably the first to examine this problem was Bodlaender, who in 1989 showed that the question whether a graph G covers graph H is at least as hard as the graph isomorphism problem [5]. As response to his work, Abello, Fellows and Stillwell studied, how hard it is for a given graph to decide, whether some other graph does cover it or not. Within two years they proved that for some graphs it is solvable in polynomial time while for other NP-complete. Later many people tried to fully characterize the complexity of this problem, but there are only partial results so far. To demonstrate that there really are both NP-complete and polynomially solvable cases, we shall show an example for each in chapter three. The first mentioned is due to Kratochvíl, Pruskurovski, Telle [14] and Fiala [7].

The last chapter deals with the complexity of regular graph covering problem. It is quite expectable that the condition of regularity could make the problem easier, which is supported by the fact that nobody so far has found a class of graphs, for which it would be NP-complete. Thereto, we have found a big one, for which it is in P. Our result is very interesting especially because the result is not dependent on the structure of the given graphs, but only on the ratios of their sizes. However, the question, whether it is possible to extend this result for a bigger class of graphs still remains open.

Chapter 1

Preliminaries

As we have promised to formulate in this work various definitions of regular covering, each using a different piece of mathematics and many terms from different mathematical fields, in the following few pages we will, for the readers convenience, write down the basic definitions from all these areas. We suggest the reader to skip them for now and later in case of precariousness return here.

1.0.1 Common knowledge of graphs

We will usually work with the common definition of graph, where a *graph* G is a couple (V, E) with a *vertex set* V and with E - a subset of $V \times V$ called the *edge set*. We often give them new, more simple names, instead of writing them as ordered pairs. To avoid misunderstanding, we will sometimes denote the vertex and the edge set $V(G)$ and $E(G)$ to point out we are talking about the vertex and edge set of the graph G.

When talking about the size of a graph, we will always mean the size of its vertex set.

Below are listed the most common terms related to graphs:

Definition 1.0.1. *Neighborhood* N_v of a vertex v of a graph G is the set of all vertices u such that $(v, u) \in E$.

Definition 1.0.2. Automorphism of a graph G is a mapping $a : G \to G$ which keeps incidence.

The automorphisms of a graph form a group, called *automorphism group*, denoted usually $Aut(G)$.

Definition 1.0.3. *Path* in a graph G is a sequence of vertices and edges:

$$v_0, e_0, v_1, e_1, \dots e_{k-1}, v_k$$

such that e_i is the edge joining vertices v_i and v_{i+1} for all $i = 0, \dots k-1$.

Walk is a path such that no vertex is visited twice.

Definition 1.0.4. Joining the beginning and the end vertex of a path creates a *cycle*.

Definition 1.0.5. *Tree* is a connected graph without cycles.

There is a number of equivalent conditions for a graph to be a tree. We introduce them in the following theorem:

Theorem 1.0.6. *For a graph G the following are equivalent:*

- *G is a tree,*
- *for all $u, v \in V(G)$ there exists exactly one path from u to v,*
- *G is connected, but by removing any of its edges it becomes disconnected,*
- *G has no cycle, but adding an edge joining any two vertices creates a cycle,*
- *G is connected and has $|V(G)| - 1$ edges.*

Definition 1.0.7. Let G be a graph. *Spanning tree* T is the biggest subgraph that contains all vertices, but no cycles, precisely: T is a tree with vertex set $V(T) = V(G)$ and edge set $E(T) \subseteq E(G)$.

Note that by fixing one spanning tree T of G, one can create cycles by adding the other edges - the so called *fundamental cycles*. All cycles of G can be created by disjoint union of these cycles.

Sometimes we will use a more knotty definition of graphs as quadruples.

Definition 1.0.8. *Graph* is a quadruple (D, V, I, λ), where:

- D is a set of darts,
- $V \neq \emptyset$ is a set of vertices,
- $I : D \to V$ is an incidence function, which assigns each dart its initial vertex,
- λ is an involution on D, which interchanges the dart's initial and terminal vertex.

This definition will be useful in several cases, and it will help us create a few very nice graph coverings. However, if we don't explicitly say that we are using it, then we talk just about ordinary graphs.

1.0.2 Some group terms

We hope everyone knows what a group, in mathematical sense, is. However, we will write here the terms related to this topic, which will be used thereinafter.

Definition 1.0.9. A subgroup B of a group A is said to be *normal*, if holds that $aB = Ba$ for each $a \in A$.

There are many equivalent conditions for subgroup normality, but for our purposes this condition will be sufficient.

Definition 1.0.10. A *center* of a group A is a subgroup containing all elements that commute with each element of A.

Definition 1.0.11. *Order* of a group A is the number of its elements.

Order of an element a in the group A is the smallest $k \in \mathbb{N}$ such that $a^k = 1_A$, where 1_A is the neutral element of the group.

Definition 1.0.12. An *action* of a group A on a set X is a map $\phi : A \times X \to X$ that fulfills:

- if 0 is the neutral element of A, then $\phi(0, x) = x$ for all $x \in X$, and
- $\phi(a, \phi(b, x)) = \phi(ab, x)$ for all $a, b \in A$.

Stabilizer St_x of an element $x \in X$ is a set

$$St_x = \{a \in A, \phi(a, x) = x\}.$$

Stabilizer of set X is a set

$$St = \bigcap_{x \in X} St_x.$$

Orbit of an element $x \in X$ is the set

$$O_x = \{y \in X, \exists a \in A : \phi(a, x) = y\}.$$

If there is only one orbit, we say ϕ is *transitive.*

For all $a \in A$ we will denote fix_a the set of all elements that are fixed by a, i.e.

$$fix_a = \{x \in X, \phi(a, x) = x\}.$$

Note that the stabilizer of each element as well as the stabilizer of the group are subgroups of the acting group.

An essential finding about group actions is picked up in the following lemma, called Burnside's lemma, although its real authors are Cauchy and Frobenius.

Lemma 1.0.13. *Let X be a set and A a group acting on it. Then:*

$$\textstyle\sum_{O_a} 1 = \frac{1}{|A|} \sum_{a \in A} |fix_a|.$$

A special case of group action is when the set is the base set of the group acting. It is then called translation, and because the group does not have to be Abelian, there exist both left and right translations. More exactly:

Definition 1.0.14. *Right translation* is a mapping $\lambda_a : a \times A \to A$, defined $\lambda_a(b) = b \circ a$ where $a, b \in A$. *Right translation group* is the set of all translations $\{\lambda_a, a \in A\}$, composition defined as $\lambda_a \circ \lambda_b = \lambda_{a \circ b}$.

Left translation and *left translation group* are defined analogically.

Note that translation group really is a group.

1.0.3 A piece of topology

As the concept of coverings has roots in topology, we feel it is our duty, before defining graph covering, to put down also the definition of covering topological spaces in general. Looking at graphs as at topological spaces will also help us formulate an equivalent condition for covering regularity. For this purpose we put here up a few common definitions and theorems from algebraic topology.

Definition 1.0.15. *Topological space* is a set X with a collection of its subsets $\mathcal{G} = \{T_i, i \in I\}$ where I is a index set, which fulfill these four conditions:

- $\emptyset \in \mathcal{G}$,
- $X \in \mathcal{T}$,
- $\bigcap_{i \in I} T_i \in \mathcal{G}$, where $T_i \in \mathcal{G}$ for I finite, and
- $\bigcup_{i \in I} T_i \in \mathcal{G}$, where $T_i \in \mathcal{G}$ for all I.

There are of course many different types of topological spaces. We will work with these:

Definition 1.0.16. A *pointed topological space* is a topological space with a given *base point*. It is usually denoted as $(\mathcal{G}, g)$, where g is the base point.

Definition 1.0.17. Let x and y be two points in $\mathcal{G}$. A *path* from x to y is a function $f : [0,1] \mapsto \mathcal{G}$ such that $f(0) = x$ and $f(1) = y$. Topological space in which there is a path joining each two of its points is called *path-connected*.

And of course, we need to define some relation between the spaces themselves, and also between their components. The first can be done via homeomorphism, and the latter by homotopy equivalence.

Definition 1.0.18. Topological spaces $\mathcal{G}$ and $\mathcal{H}$ with base sets X and X' are said to be *homeomorphic* if there exists a mapping $f : X \mapsto X'$ such that:

- f is a bijection,
- f is continuous,
- f^{-1} is continuous.

Definition 1.0.19. Given two topological spaces $\mathcal{G}$ and $\mathcal{H}$, mappings $f_0,\ f_1 :\ \mathcal{G} \mapsto\ \mathcal{H}$ are said to be *homotopy equivalent*, if there exists a function $F : \mathcal{G} \times [0,1] \mapsto \mathcal{H}$ such that

- $F(\mathcal{G}, 0) = f_0$,
- $F(\mathcal{G}, 1) = f_1$,
- $F(\mathcal{G}, t)$ is continuous for all $t \in [0,1]$.

For better comprehension see image 1.1.

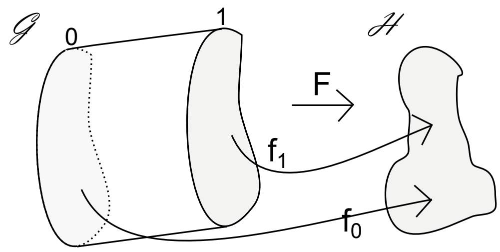

Figure 1.1: Homotopy.

Definition 1.0.20. We say that topological spaces are *homotopy equivalent*, if there exist mappings $f : \mathcal{G} \mapsto \mathcal{H}$ and $g : \mathcal{H} \mapsto \mathcal{G}$ such that for their compositions holds that $g \circ f = id_{\mathcal{G}}$ and $f \circ g = id_{\mathcal{H}}$.

Useful characteristic of a pointed topological space is its fundamental group $\pi_1(\mathcal{G}, g)$. It is a group, whose elements are classes of homotopy equivalent loops beginning at the base point of the space. Although it is a specific case of what was mentioned in the previous definition, let us express, what are loops in topological spaces, and when are two loops of a pointed topological space homotopy equivalent. Informally, if they can be continuously deformed one to another. And formally:

Definition 1.0.21. Let $(\mathcal{G}, g)$ be a pointed topological space. A *loop* with base point g is a continuous function $l_0 : [0,1] \mapsto \mathcal{G}$ such that $l_0(0) = l_0(1) = g$. Two such loops l_0 and l_1 are called homotopy equivalent ($l_0 \sim l_1$) if there exists a mapping $h : [0,1] \times [0,1] \mapsto \mathcal{G}$ fulfilling that $h(t,0) = l_0(t), h(t,1) = l_1(t)$ and $h(0,t) = h(1,t) = g$ for all $t \in [0,1]$. Mapping h is called homotopy from l_0 to l_1.

It is quite obvious that homotopy is an equivalence relation. Therefore we may talk about classes of homotopy equivalent loops. Composition of two loops is defined so that we run along both of the loops in double speed. Composition defined like this is apparently homotopy associative (i.e. $(l_3 \circ l_2) \circ l_1 \sim l_3 \circ (l_2 \circ l_1)$). Neutral element will be the class of the constant map $l : [0,1] \mapsto g$. The inverse element of a class of a loop $l(t)$ is the class of the loop $l(1-t)$. Thanks to this we may define the fundamental group of a pointed topological space as:

Definition 1.0.22. *Fundamental group* $\pi_1(\mathcal{G}, g)$ of a pointed topological space $(\mathcal{G}, g)$ is a group, whose elements are classes of homotopy equivalent loops based in g, with group operation defined as:

$$l_2 \circ l_1 = \begin{cases} l_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ l_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is not difficult to realize that the fundamental group really *is* a group.

We see that the fundamental group $\pi_1(\mathcal{G}, g)$ depends on the choice of the base point g. A natural question that arises, is: what is the condition that the base points g_1 and g_2 have to fulfill for the fundamental groups $\pi_1(\mathcal{G}, g_1)$ and $\pi_1(\mathcal{G}, g_2)$ to coincide?

Theorem 1.0.23. *Let $\mathcal{G}$ be a topological space, g_1, g_2 points in $\mathcal{G}$. If there exists a path from g_1 to g_2, then the fundamental groups $\pi_1(\mathcal{G}, g_1)$ and $\pi_1(\mathcal{G}, g_2)$ are isomorphic.*

Two distinct paths α, β from g_1 to g_2 induce the same isomorphism of the concerned fundamental groups if and only if $\beta^{-1} \circ \alpha$ is from the center of $\pi_1(\mathcal{G}, g_1)$

Proof. Let us take a path α from g_1 to g_2 and map the loop l based in g_1 onto the loop $\alpha \circ l \circ \alpha^{-1}$ based in g_2. This mapping induces the isomorphism of the considered fundamental groups.

Now let us take some $[l] \in \pi_1(\mathcal{G}, g_2)$ and map it with the isomorphism induced by α. Then:

$$\alpha \circ [l] \circ \alpha^{-1} = [\alpha \circ l \circ \alpha^{-1}] = [\beta \circ l \circ \beta^{-1}] = \beta \circ [l] \circ \beta^{-1}$$

Therefore

$$\beta^{-1} \circ \alpha \circ [l] \circ \alpha^{-1} \circ \beta = [l],\ \beta^{-1} \circ \alpha \circ [l] = [l] \circ \beta^{-1} \circ \alpha$$

i. e. $\beta^{-1} \circ \alpha$ commutes with $[l]$. □

Consequently, path-connected topological space has (up to isomorphism) only one fundamental group.

Given two pointed topological spaces and a mapping from one to the other, this mapping induces a mapping of their fundamental groups.

Definition 1.0.24. Let $(\mathcal{G}_1, g_1)$ and $(\mathcal{G}_2, g_2)$ be two pointed topological spaces and f a morphism of these spaces, which maps g_1 to the base point g_2 of $\mathcal{G}_1$. We define

$$f_* : \pi_1(\mathcal{G}_1, g_1) \mapsto \pi_1(\mathcal{G}_2, g_2)$$

so that for the class of the loop $\omega(t)$ beginning at the point g_1 holds:

$$f_*[\omega(t)] = [f\omega(t)].$$

We should discuss, whether f_* is a group homomorphism, in other words, whether $(f_2 \circ f_1)_* = f_{2*} \circ f_{1*}$.

Let us have maps $f_1 : (\mathcal{G}_1, g_1) \mapsto (\mathcal{G}_2, g_2)$ and $f_2 : (\mathcal{G}_2, g_2) \mapsto (\mathcal{G}_3, g_3)$ and let $f_1(g_1) = g_2$ and $f_2(g_2) = g_3$. Further, let f_{1*} and f_{2*} be the maps induced by them. Then

$$(f_2 \circ f_1)_*[\omega(t)] = [f_2 \circ f_1\omega(t)] = f_{2*} \circ f_{1*}[\omega(t)],$$

and we can conclude that it really is a group homomorphism.

Chapter 2

Structure of graph coverings

Graph covering is just a special type of covering of topological spaces. Therefore we introduce first the definition of covering of topological spaces. Nevertheless, it will be useful when we proceed to another part of this chapter... If someone feels precariously about topological spaces, he can find their exact definition, as well as some of their properties, among preliminaries.

Definition 2.0.25. Covering. Given two path-connected topological spaces $\mathcal{G}$ and $\mathcal{H}$, mapping $p : \mathcal{G} \to \mathcal{H}$ is said to be a *covering* of $\mathcal{H}$, if for every $x \in \mathcal{H}$ there exists its neighborhood N_x such that $p^{-1}(N_x)$ is homeomorphic to $N_x \times D$, where D is a discreet set and $p|_{p^{-1}(N_x)}$ coincides with the natural projection $\pi : N_x \times D \to N_x$.

The set of vertices $p^{-1}(x)$ is called the *fiber* of x, and its elements are called *lifts* of x.

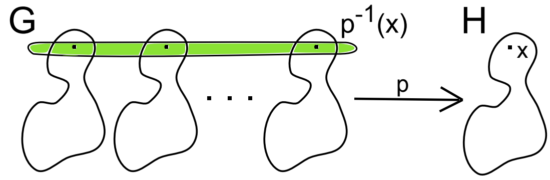

Figure 2.1: Covering

Remark 2.0.26. Fibers of all points have the same cardinality. (If not, then there would be some neighborhoods N_x and N_y of points x and y of different cardinality, such that $N_x \cap N_y \neq \emptyset$. But how many copies of the element from the intersection should there be?) Therefore, if the fibers have n elements, we can say we have an $n - fold$ covering.

Each graph can be viewed as a topological space. Its set is the set of its vertices together with the edges, thought of as homeomorphic images of the unit interval $(0, 1)$. When we are aware of this fact, following definition of graph covering is just a refinement of the definition above.

Definition 2.0.27. Covering - graphs. Given two graphs G, H and $p : G \to H$ we say p is a *covering* of H if it is surjective and a local isomorphism, that means, it maps the neighborhood of a vertex $g \in G$ bijectively to the neighborhood of $p(g) \in H$.

Remark 2.0.28. Note that the number of vertices of G always has to be a multiple of the number of vertices of H, and that the degree of all vertices in the fiber of each $v \in H$ has to be the same as the degree of v itself.

As non-connected graphs can be thought of as groupings of connected graphs, we will constrict our attention to connected graphs only.

2.1 A few examples of coverings

There will be many pictures in this section, because we will present here some examples of graph coverings. In addition to this, we will have a look at two basic questions one may ask, when talking about coverings:

- "What are the covers of a given graph?",
- "What graphs does a given graph cover?",

and try to answer these questions for graph H showed in image 2.2.

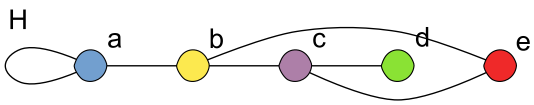

Figure 2.2: Graph H.

First we will try to find some of its covers. The most simple one is the graph itself. The covering projection will be the identity function, and everything will work nicely. If we want to make more-fold covers, the best way is to find a spanning tree of H, make the desired number of copies of it, and then, for an edge (u, v) join some of the pre-images of u with an arbitrary pre-image of v. One of the 3-fold covers is in picture 2.3.

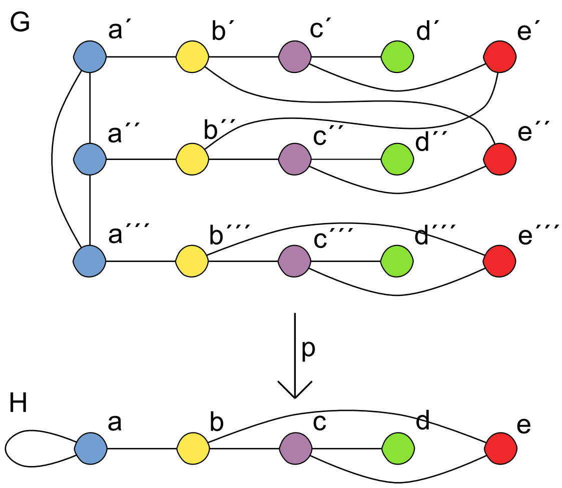

Figure 2.3: Graph covering p.

The second question is in this case very boring. We see that there is only one vertex of degree one in H, hence it is the only vertex of the pre-image of the vertex of degree one of the covered graph - the covering projection has to be 1-fold, i.e. it is the identity and the graph covers only itself.

Very nice example of a multi-covered graph is K_4 - complete graph on four vertices (see figure 2.4). What are its covers? They, of course, have to have $n \times 4$ vertices, and because K_4 is three-regular, the covering graph has to be, too.

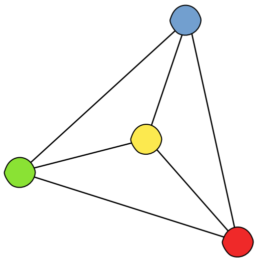

Figure 2.4: K4.

For $n = 2$ we are looking for a cubic graph on 8 vertices. There are many of them, therefore let us restrict on the vertex transitive ones: two of them are covers of K_4. One might suppose that vertex transitivity is a strong presumption and that all vertex transitive graphs on 8 vertices in fact should be covers of K_4. But this turns up to be mis-judgment, as proves the third graph in image 2.5.

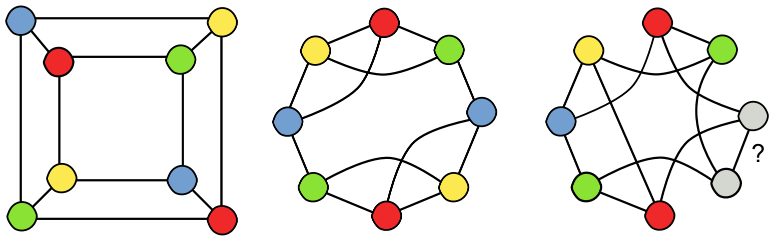

Figure 2.5: Cubic vertex transitive graphs on 8 vertices: Q_3, some unnamed graph and the Mobius ladder.

An example of a multi-covering graph on not too many vertices is the already mentioned Q_3. Except for K_4, it can be created from four copies of a three-regular graph on just two vertices. Many more options supervene, when we take the definition 1.0.8 of a graph using darts. Rather than describing them, look at figure 2.6.

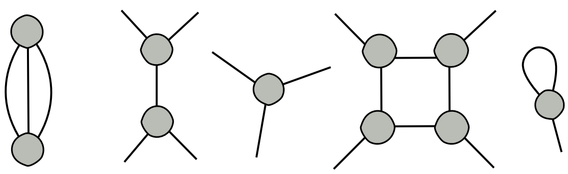

Figure 2.6: Quotients of Q_3.

There are many other nice examples of coverings, and there is much to be said about coverings in general. However we will go no further in this direction, instead we will advance to the regular ones.

Chapter 3

Structure of regular graph coverings

In this chapter we will present a special type of covering - regular covering. As it was already mentioned, there are different approaches to this concept. Each of the sections bellow follows a different one of them.

3.1 Regular coverings & graph theory

The easiest definition, which does not need too much explanation, is in terms of graph theory.

Definition 3.1.1. Regular covering. Let G and H be graphs, $p : G \to H$ a covering. p is said to be *regular*, if all lifts of any closed chain in H are closed or non-closed in G simultaneously.

It's quite straightforward that each two-fold covering is regular.

Problem 1. Let us look once more at the covering in picture 3.1. Is this covering regular?

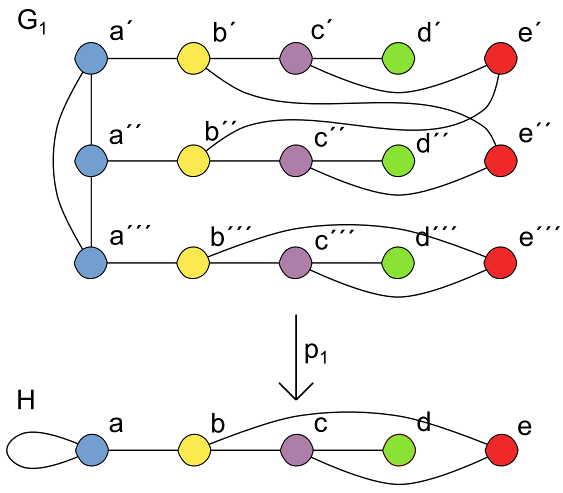

Figure 3.1: Example of a regular covering.

If we take the lifts $b''c''e''b'$ and $b'''c'''e'''b'''$ of the closed chain $bceb$, we see that the first of them is not closed, but the second one is. Therefore the covering is not regular, according to definition 3.1.1.

The same graph can also be covered regularly. A candidate is in figure 3.2.

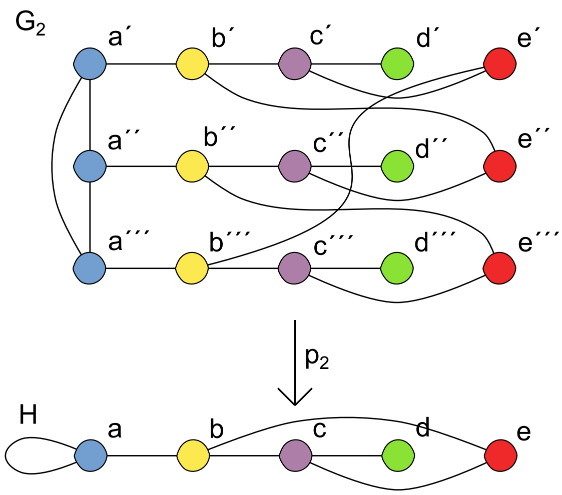

Figure 3.2: Example of a regular covering.

Problem 2. We ask: Is covering in figure 3.2 regular? To prove that it is regular, we should check lifts of all cycles from H in G. The graph is symmetric, therefore we can choose just one of the lifts of paths aa, $cebc$ and $cebaac$. It is always non-closed, and so is walking along the paths twice. However, walking along them three times has all lifts closed. The covering is regular.

An important question is - given two graphs, one cover of the other, what does covering regularity depend on? Just on the structure of the graph? What role does play the covering projection? In picture 3.3 there is an example of a graph G that covers the same graph H both regularly and irregularly. Regularly via covering p_1: lifts of the blue edge are non-closed, and remain so, as well as the lifts of the edge being walked along four times - then they are all closed. Checking all possible walks, using both edges in arbitrary order, we would prove that the covering really is regular. On the other hand, covering p_2 is irregular, which can be seen for example from the lifts $a'b'$ and $a''b'$ of path ab pointed out in figure 3.4.

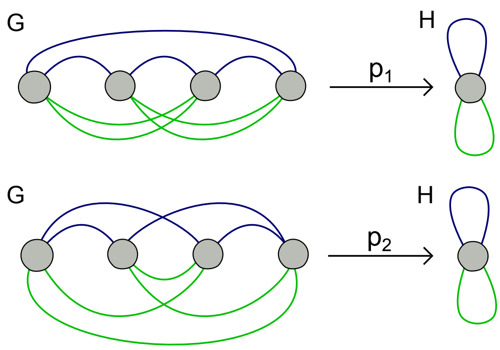

Figure 3.3: Regular and irregular covering between the same pair of graphs.

Therefore the answer is: Regularity does not depend just on the structure of G and H. Covering projection matters!

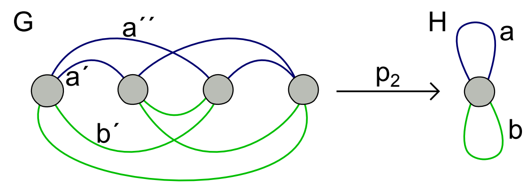

Figure 3.4: Why p_2 is not regular.

3.2 Regular coverings & automorphism groups

An equivalent condition for graph regularity can be expressed in terms of groups and group actions. Why group actions? Taking a group of automorphisms of a graph, it acts on the vertex set of the graph. And so does each its subgroup. In this section we will look at the action of its special subgroup, called the covering transformation group. Therefore let us write, which automorphisms are in this subgroup:

Definition 3.2.1. If G, H are graphs, $p : G \to H$ a covering, then the *covering transformation group* is a subgroup of the group of automorphisms of G, denoted $CT(p) = \{\phi \in Aut(G), p \circ \phi = p\}$.

In other words, in the covering transformation group are those automorphisms for which the following diagram commutes.

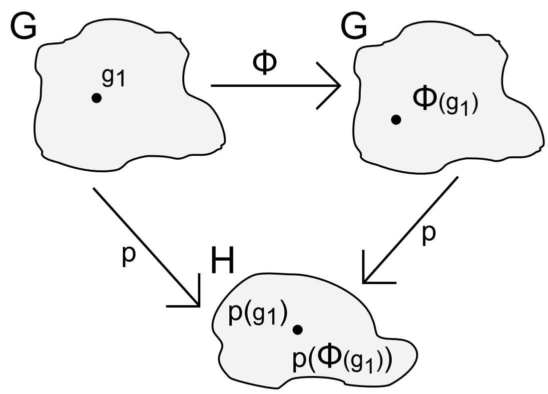

Figure 3.5: Images of $g_1 \in G$ when mapped by p, ϕ and $p \circ \phi$.

Remark 3.2.2. A quick inspection will show that these automorphisms really do form a group, so the name "covering transformation group" is well-deserved. Another useful observation is that they fix fibers (they are automorphisms of each fiber itself). However, they don't fix any vertices (except from the identity automorphism, of course). Therefore we get that $|CT(p)| \leq |p^{-1}(h)|$ for any $h \in H$.

Natural question arises: when does equality hold in this inequality?

Theorem 3.2.3. Covering regularity. *Covering $p : G \to H$ is regular, if and only if G has automorphisms that are vertex transitive on each fiber, i.e. when $|CT(p)| = |p^{-1}(h)|$ for all $h \in H$..*

Proof. It is quite obvious that if there was a closed path some of whose lift was closed and other non-closed, then the vertices of those paths could not be mapped on each other by any automorphism of G and so the equality could not hold.

Now suppose we have a covering p that is regular according to definition 3.1.1. That means it fulfills the condition that every lift of any closed chain is closed or non-closed if and only if all other lifts of the same chain are closed or non-closed. We want to prove that for any vertex $h \in H$ and vertices $g_1, g_2 \in p^{-1}(h)$ there exists an automorphism a of G that maps g_1 onto g_2.

Let us take a spanning tree T of H and construct all lifts of T. We define $a(g_1) = g_2$ and map the lift of T containing $g_1(T_1)$ onto the lift of T containing $g_2(T_2)$. Then let us take e - one of the edges that are not in T, with end vertices s and r. It's lifts are either all within some of the trees T_i or all join two different copies T_i and T_j. If the first case supervises, the automorphism preserves this edge. If the second, we take the lift that begins at T_1 and ends at some T_k. This edge corresponds to exactly one edge beginning at $a(s)$ and ending at some T_q. Now we can continue defining a. We do it so that T_j is mapped onto T_q. Then we will look at the lift of this edge that begins in j-th liar and so on... This defines an automorphism of H.

The question is, if we do the same for some other edge, will it be consistent with the previous definition of a? Thanks to the presumption that every lift of any closed chain is closed or non-closed if and only if all other lifts of the same chain are closed or non-closed, it has to be. $\square$

Now let us look at the two problems from the previous section and try to solve it via theorem 3.2.3.

Problem 1. Is covering $p_1 : G_1 \to H$ from 3.1 regular? If the covering was regular, there should be an isomorphism *izo* which would map vertex d' onto vertex d''', as shown in figure 3.6. Then the images of c', b' and a' are clear. However, the other neighbor of b' is e'', unlike the neighbor of c', which is e'. Thus the fiber of d is not vertex transitive and the covering is not regular.

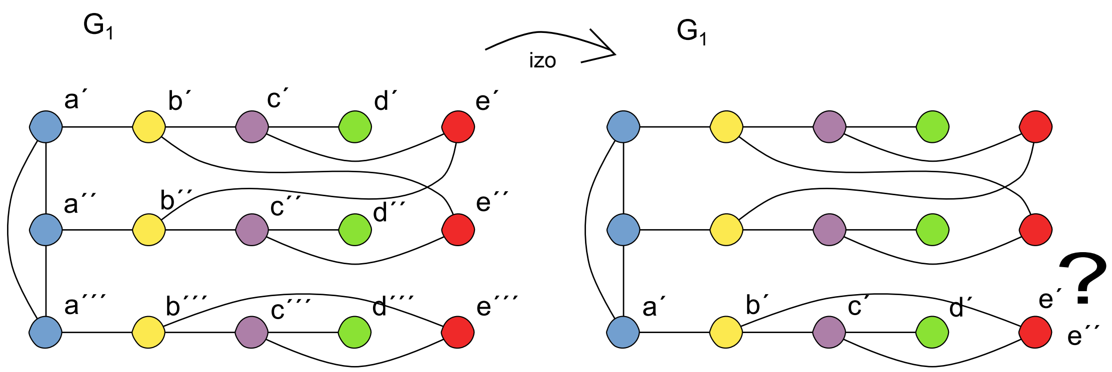

Figure 3.6: Automorphism mapping d' onto d'''

Problem 2. Does G_2 from figure 3.2 cover H regularly? We would have to check vertex-transitivity of the fibers... The graph is highly symmetric, so we would only have to check five pairs of vertices. As all the non-spanning tree edges are joining two different copies of the spanning tree, there won't arise any ambiguity as in the non-regular case. Just for illustration, we show the automorphism that maps d' onto d''' in picture 3.7.

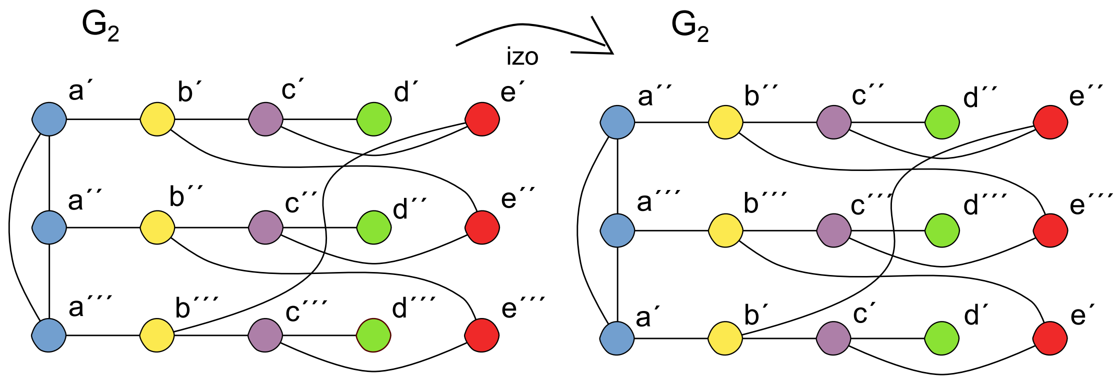

Figure 3.7: Automorphism mapping d' onto d'''

3.3 Regular coverings & algebraic topology

In this section we shall try to deal with the term of covering regularity with a little help of algebraic topology. We will look at graphs as at pointed topological spaces, therefore let us first distinguish a point g of a graph G. Then a loop based in g is a path in G whose both end vertices are g. Two fundamental groups of a graph are isomorphic if there is a path between the given base vertices. As we will consider only connected graphs (connected graphs are path-connected topological spaces), their fundamental groups won't depend on the choice of the base point (theorem 1.0.23 in the preliminaries).

Theorem 3.3.1. Covering regularity. *Let G and H be graphs, $p : G \to H$ a covering. We say that p is regular, if and only if the fundamental group $p_*\pi_1(p(G))$ is a normal subgroup of $\pi_1(H)$, where p_* is the induced mapping of fundamental groups defined in 1.0.24.*

Proof. Let us first choose a base point g of G. Covering p maps it onto $p(g) \in H$. We will set it as the base point of H. Let $p_* : \pi_1(G, g) \mapsto \pi_1(H, p(g))$ be the mapping defined in 1.0.24. It is obvious that $p_*(\pi_1(G, g))$ is a subgroup in $\pi_1(H, p(g))$ and a class of a loop $[\omega(t)]$ is an element of this subgroup if and only if ω begins at g and is closed in G.

Now let us take two arbitrary pre-images g_1 and g_2 of $p(g)$. We have already mentioned that for two different base points g_1 and g_2 the groups $p_*(\pi_1(G, g_1))$ and $p_*(\pi_1(G, g_2))$ do not have to coincide. However, it holds that:

$$p_*(\pi_1(G, g_2)) = \alpha^{-1} p_*(\pi_1(G, g_1))\alpha, \tag{3.1}$$

where α is a path from g_1 to g_2 in G. This equality holds if and only if $p_*(\pi_1(G, g_1))$ is a normal subgroup of $\pi_1(H, p(g))$. We may also notice that the two subgroups are identical if all lifts of the paths in H beginning at g_1 and g_2 are closed or non-closed simultaneously, thus covering p is regular. $\square$

In the previous sections we studied the non-regular covering of graphs from image 2.3. Now let us show it really is not regular using theorem 3.3.1.

Problem 1. First we ask, whether G_1 covers H regularly. Fundamental group of a graph is always induced by chord-less cycles. They are indicated by different colors in picture 3.8. The image of the green loop beginning at a' is: $\beta\gamma\delta\epsilon\gamma\delta\epsilon\beta^{-1}$. The image of the loop beginning at a''', in picture drawn in red, is: $\beta\gamma\delta\epsilon\beta^{-1}$. When we want to move it, to begin at a', too, we obtain: $\alpha\alpha\beta\gamma\delta\epsilon\beta^{-1}\alpha^{-1}\alpha^{-1}$. But these two sequences do not represent the same loops, and they are not homotopic loops, either. The first one runs around the cycle twice, while the second one just once. Therefore the covering is not regular.

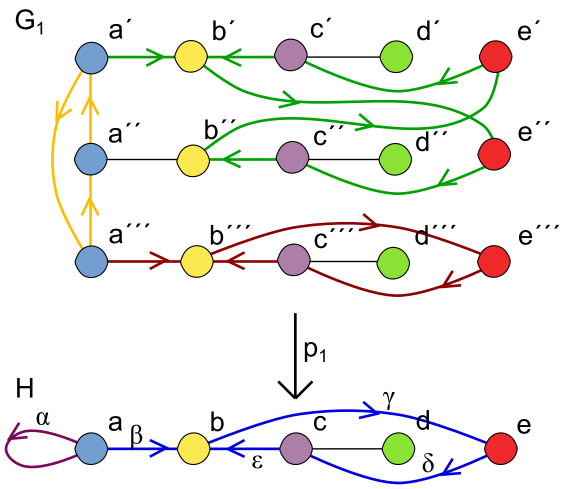

Figure 3.8: Loops in the big graph and their image in the small one.

Problem 2. Does G_2 from figure 3.2 cover H regularly? The chord-less cycles look as indicated in picture 3.9. We won't write the proof formally, let us just discuss what happens to the loops when being mapped by this projection. It is quite easy to see that there cannot occur such situation as in the previous case. All images of the green loop run along cycle $\gamma\delta\epsilon$ three times, no matter at which vertex we begin. And it is similar with the image of the yellow cycle.

3.4 Regular coverings & voltage graphs

Another way to study graphs is via voltage assignments. And, as we will show, they will have to do also with graph coverings and another equivalent condition for regular graphs will be in terms of voltages. Moreover, they will be useful for us when dealing with complexity of graph covering regularity in the last chapter of this work.

First we will define voltage assignment and the way of deriving from a given graph and a voltage assignment a new graph - the so called voltage graph.

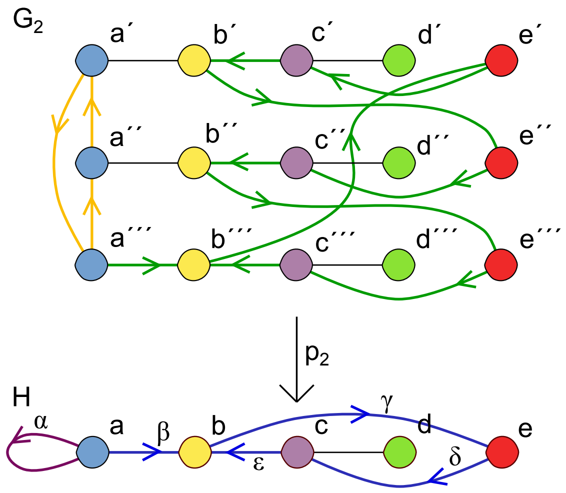

Figure 3.9: Loops in the big graph and their image in the small one.

We will denote e^+ some orientation of an edge e and e^- the opposite orientation of the same edge, and for a given graph B, $\bar{E}$ will indicate the set of all orientations of all edges of that graph, i.e. $\bar{E} = \{e^+, e^- | e \in E(B)\}$.

Definition 3.4.1. Let H be a connected graph, A a group. Mapping $\mu : \bar{E} \to A$ such that $\mu(e^+) = (\mu(e^-))^{-1}$, will be called an *ordinary voltage assignment* on the graph H.

An *ordinary voltage graph* then is a graph G constructed in the following manner:

- $V(G) = V(H) \times A$,
- (e, a) is an edge from (u, a) to $(v, \mu(e) \circ a)$, derived by an edge $e = (u, v)$, and an element $a \in A$, and
- $E(G) = E(H) \times A$.

For easier comprehension, let us look at an example. Let us take graph H from image 3.10, and for A we shall take the additive group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Thus the voltage graph G will consist of four copies of H, tagged by the elements of $\mathbb{Z}_4$. We can assign the edges of H any elements of the group. So as not to get totally scrawled picture, we will assign most of the edges the neutral element and only two of them will be assigned non-trivially, by 1 and 2. Now we only need to draw the edges of G, and this will be done by a simple rule: if the corresponding edge of H was assigned id, the lifts of this edge will remain within each copy of H. Now, the edge assigned 1 will go from the red vertex in the i-th layer to the green vertex in the $(i+1)$-th layer. The same will be done for the edge assigned by 2.

A permutation voltage assignment is almost the same as an ordinary voltage assignment, only: you take a symmetric group S_n, and the derived graph is not as big as the group, but only as big as the number of the elements permuted - n. For completeness:

Definition 3.4.2. Let H be a connected graph, S_n a symmetric group. Mapping $\mu : \bar{E} \to S_n$ such that $\mu(e^+) = (\mu(e^-))^{-1}$ is called a *permutation voltage assignment* on the graph H.

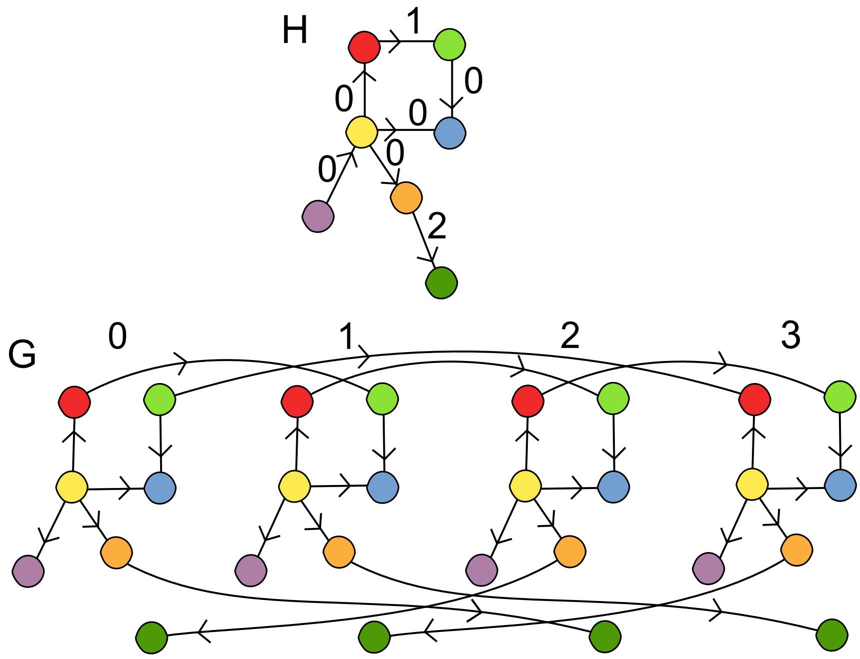

Figure 3.10: An ordinary voltage graph G derived from H using $\mathbb{Z}_4$.

A *permutation voltage graph* then is a graph G constructed in the following manner:

- $V(G) = V(H) \times \{1, 2, \dots n\}$,
- (e, i) is an edge from (u, i) to $(v, (\mu(e))(i))$ derived by an edge $e = (u, v)$ and an element $i \in \{1, 2, \dots n\}$, and
- $E(G) = E(H) \times \{1, 2, \dots n\}$.

Worthy of remark is that each ordinary voltage assignment can be viewed as a permutation voltage assignment. One can do it very easily by taking the left or right translation group as the permutation group. The exact definition of translation groups is in 1.0.14.

Let us look at the example in picture 3.10. Instead of taking group $\mathbb{Z}_4$ we will take the symmetric group S_4. The edges will then be reassigned so as you can see in picture 3.11. Permutation voltage graph will be exactly the same as the beginning ordinary voltage graph.

Remark 3.4.3. Natural projection $p : G \to H$ which forgets the second coordinate apparently is a graph covering.

Remark 3.4.4. The covering created from an ordinary voltage assignment is always a regular one. It can be seen from the group structure, because a closed path $e_1, e_2, \dots e_k$ beginning at some vertex $u \in H$ is lifted onto paths beginning at (u, a) and ending at $(u, a \circ \mu(e_1) \circ \dots \circ \mu(e_k))$ for all $a \in A$. If $a \circ \mu(e_1) \circ \dots \circ \mu(e_k) = a$ for some $a \in A$, we can cancel the a out and obtain $\mu(e_1) \circ \dots \circ \mu(e_k) = 1$, and therefore all other lifts will be closed, too.

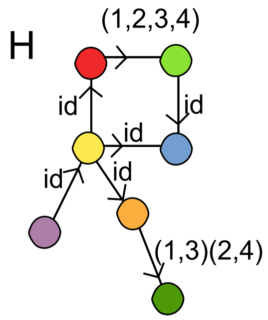

Figure 3.11: A permutation voltage assignment derived from ordinary voltage assignment in figure 3.10.

Constructing the voltage graph may be pretty messy. For better lucidity, we can use a simple trick:

Remark 3.4.5. Let us suppose we have a graph H and a group A and we have already labeled the edges of the graph with some elements of the group, we shall call this mapping μ. Now let us take an arbitrary (but fixed) spanning tree T of the graph and create a new mapping $\bar{\mu}$ so that

$$\bar{\mu}(e) = 1_A$$

for all $e \in T$. Adding any of the other edges to T results in creating a cycle. We shall label these edges with the composition of all the elements of A assigned by μ to the edges of the cycle (in the proper order!), i.e. if adding an edge f resulted in creating cycle $f \circ e_1 \circ e_2 \circ \ldots \circ e_k$, then

$$\bar{\mu}(f) = \mu(f) \circ \mu(e_1) \circ \mu(e_2) \circ \ldots \circ \mu(e_k).$$

Note that the homomorphism μ_* remains across this reassigning preserved. Consequently, voltage graph obtained after this relabeling is isomorphic to the original one.

Graph in picture 3.10 is almost relabeled. There is only one non-trivially assigned edge in the cycle, only the edge from the orange vertex to the green vertex should be also assigned 0.

This work is dedicated to regular covering. We have already mentioned that coverings derived from ordinary voltage assignments are always regular. But, how about permutation voltage assignments? Are they regular? Under what condition? The answer is very simple:

Theorem 3.4.6. *Let H be a graph, S_n a symmetric group and G their voltage graph. Let S be the subgroup generated by the elements of S_n assigned to the edges of H. Then covering $p : G \to H$ defined as the natural projection is regular if and only if none of the elements of S, except for the identity element has a fixed point.*

Remark 3.4.7. Permutation groups, which fulfill the condition of not having any permutation that would fix some point, except from the identity, are also called regular permutation groups.

Proof. To prove this theorem, we have to think of what a fixed point of a permutation is. Let us have a permutation which fixes some point (copy of H), but is not identity. Then the fixed point induces a cycle within one copy of H in the voltage graph G, but the other, non-fixed points induce a bigger cycle, which runs through more copies of H. The small cycle is a closed lift of the closed path in H, but the bigger cycle breaks up into a number of non-closed lifts of the closed path, hence the covering is not regular.

For the straightforward implication, just retrace these arguments... □

Now the time has come, to look again at problems 1 and 2 and answer the question of regularity of coverings p_1 and p_2 using this theorem.

Problem 1. We inquire about regularity of covering p_1 in image 3.12. One can see the spanning tree of H in red and its three copies. The permutations assigned to the non-spanning tree edges (a, a) and (b, e) will be (123) and $(12)(3)$ respectively. The second permutation has a fixed point, i.e. the covering is not regular.

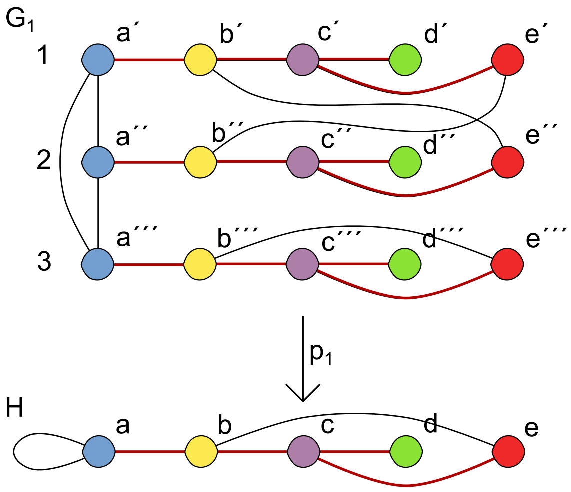

Figure 3.12: Spanning tree of H and its 3 copies in G_1.

Problem 2. Is covering p_2 from figure 3.13 regular? The permutations that should be assigned to the edges of H are (123) and (123). They generate only cyclic permutations (132) and the identity. Thus covering p_2 is regular.

In the previous we have defined different groups allied somehow to the term of covering, which may be a bit confusing. We know that the definition of covering regularity via each of them is equivalent to the others... However, one might ask, how do these groups themselves (not just the coverings related to them) bear on. In the following we shall make good this omission.

Firstly, voltage assignments induce a homomorphism of the fundamental group of the graph and the group given:

Theorem 3.4.8. *Let $p : G \mapsto H$ be a covering. Then the voltage group A associated to p and the fundamental group $\pi_1(H, v)$ are homomorphic.*

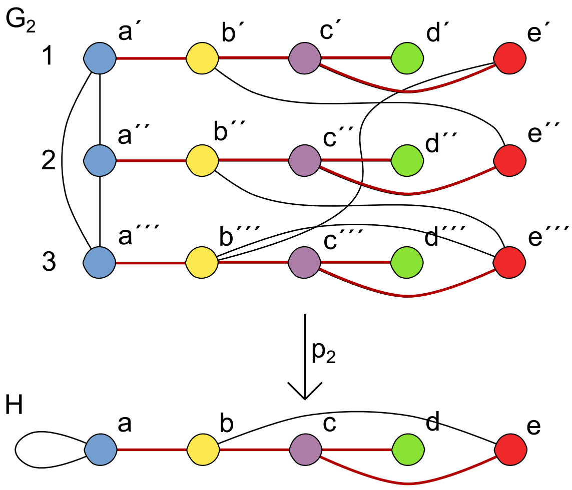

Figure 3.13: Spanning tree of H and its 3 copies in G_2.

Proof. Let $l = e_1, e_2, \dots e_k, e_1$ be a loop in $\pi_1(H, v)$, μ a voltage assignment, and $\mu(e_1), \mu(e_2), \dots \mu(e_k), \mu(e_1)$ the corresponding voltages. The desired homomorphism then will be a mapping

$$\mu_* : \pi_1(H, v) \to A$$

defined as:

$$\mu_*(l) = \mu(e_1) \circ \mu(e_2) \circ \dots \circ \mu(e_k) \circ \mu(e_1).$$

This mapping surely is well defined, and a homomorphism (because to compose two loops means to string together their edges, and then omit the parts that are homotopy equivalent to a point, which is equivalent to aligning corresponding voltage assignments and omitting incurred identities). □

Furthermore, similar construction can be used to prove that:

Theorem 3.4.9. *If $p : G \mapsto H$ is a regular covering, then the covering transformation group $CT(p)$ is isomorphic to the voltage group A associated with the covering.*

Proof. We know (theorem 3.2.3) that $CT(p)$ fixes fibers, and that for each $u, v \in p^{-1}(w)$ there is exactly one automorphism that maps u onto v, we will denote it $\phi_{u,v}$. Let us fix a vertex $w \in H$ and $u \in G$, and define mapping

$$\alpha : CT(p) \mapsto A$$

as

$$\alpha(\phi_{u,v}) = \mu(e_1) \circ \mu(e_2) \circ \dots \circ \mu(e_k),$$

where $\mu(e_1), \mu(e_2), \dots \mu(e_k)$ are the voltages of the edges on the path from u to v in G.

The first question is - is this well defined? Let us take two different paths, both ending at v, and denote the copies of H, in which u and v are, i and j respectively. Let the voltage assignments of those two paths be $\mu(e_1), \mu(e_2), \dots \mu(e_k)$ and $\mu(e_1)', \mu(e_2)', \dots \mu(e_l)'$, then it has to hold:

$$\mu(e_1) \circ \mu(e_2) \circ \cdots \circ \mu(e_k)(i) = j = \mu(e_1)' \circ \mu(e_2)' \circ \cdots \circ \mu(e_l)'(i), \text{ i.e.}$$
$$\mu(e_1) \circ \mu(e_2) \circ \cdots \circ \mu(e_k) \circ (\mu(e_1)' \circ \mu(e_2)' \circ \cdots \circ \mu(e_l)')^{-1}(i) = i,$$

and because none of the elements of A except for the identity has a fixed point, it follows that

$$\mu(e_1) \circ \mu(e_2) \circ \cdots \circ \mu(e_k) \circ (\mu(e_1)' \circ \mu(e_2)' \circ \cdots \circ \mu(e_l)')^{-1} = id.$$

Now we shall move on to injectivity. If there were two automorphisms $\phi_{u,v}$ and $\phi_{u,v'}$ mapped onto the same permutation $\mu \in A$, which maps the i-th copy of H in which is u onto the copy in which is v and at the same time, in which is v'. In copy there is only one pre-image of w and therefore $v = v'$ and α is injective and from the fact that the sizes of $CT(p)$ and the voltage group are both equal to $|G|/|H|$ yields that α is a bijection.

It remains to prove is that α is a homomorphism, i.e. that

$$\alpha(\phi_{u,v}) \circ \alpha(\phi_{v,z}) = \alpha(\phi_{u,v} \circ \phi_{v,z}).$$

But that is yet not written correctly, as the we have not defined $\alpha(\phi_{v,z})$. However, we know that for each $u, v, z \in p^{-1}(w)$, holds

$$\phi_{u,v} = (\phi_{v,u})^{-1}, \text{ and also}$$
$$\phi_{u,v} \circ \phi_{v,z} = \phi_{u,z},$$

because there is always only one automorphism mapping a given vertex onto another one. Therefore, instead of writing $\phi_{v,z}$ we can write $(\phi_{u,v})^{-1} \circ \phi_{u,z}$, and so obtain:

$$\alpha(\phi_{u,v}) \circ \alpha(\phi_{v,z}) = \alpha(\phi_{u,v} \circ (\phi_{u,v})^{-1}) \circ \alpha(\phi_{u,z}) = \alpha(\phi_{u,z}) = \alpha(\phi_{u,v} \circ \phi_{v,z}).$$

□

Observe that the isomorphism does not depend on the choice of the vertex u. Choosing some other vertex w one will obtain an isomorphism, which would be the same, up to conjugacy by the composition of elements of the group assigned to the edges on the path from u to w.

3.5 Voltage space

Another concept related to voltage assignments is the concept of voltage spaces. Although hereinafter we will work mostly with permutation voltages and permutation voltage graphs, let us, for completeness, bring in also the definition of voltage space and just few of its elementary properties, and refer the reader for more detail to [15]. To be more general, we will use definition 1.0.8 of graphs as quadruples within this section.

Definition 3.5.1. Let H be a connected graph. *Voltage space* is a triple (X, A, ξ), where:

- X is a non-empty set, called the *abstract fibre*,
- A is a group acting on X, the *voltage group*, and

- ξ is a mapping from the union of the fundamental groups based in all the vertices of H (we will denote it π) to A. For a walk W in H, ξ_W is called voltage of a reduced walk W. The image of the fundamental group based at u - $\xi(\pi^u)$ - will be denoted as A^u and called *local voltage group* at the vertex u.

Given a walk $W : u \to v$, it might be interesting to look for the inner automorphism of A, which would map A^u onto A^v. And it is easy to see that the desired automorphism W^* maps x onto $\xi_W^{-1} x \xi_W$. For illustration, see image 3.14.

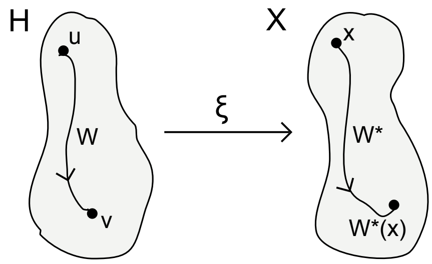

Figure 3.14: Mapping ξ.

Definition 3.5.2. We say that a voltage space is *locally transitive*, if some of the local voltage groups acts transitively on the abstract fiber.

It is obvious that if one local voltage group acts transitively on the abstract fiber, then all of them have to.

We have promised to show the connection between graph coverings and voltage spaces. This connection is called association and is defined:

Definition 3.5.3. Let G and H be graphs. Voltage space (X, A, ξ) on H is *associated* with covering projection $p : G \to H$, if there exists a labeling $l : V_G \to X$, such that for all vertices $u \in H$ holds:

- $l|_{p^{-1}(u)} \to X$ is a bijection, and
- $(l, \xi) : (G, \pi) \to (X, A)$ is a morphism of actions,

where V_G denotes the vertex set of G.

In other words, l has to fulfill:

- $l(\overline{u} \circ W) = l(\overline{u}) \circ \xi_W$ for all walks $W : u \to v$ in H and all $\overline{u} \in p^{-1}(u)$.

Note that there might be more voltage spaces associated with the same covering projection.

Theorem 3.5.4. *A covering is regular if and only if some of the local voltage groups of the associated voltage space has a normal stabilizer.*

An example of such a voltage space are Cayley voltage spaces. Cayley voltage space is a triple (A, A, ξ), where A acts on itself by right translation. Furthermore holds that:

Theorem 3.5.5. *Each regular covering can be associated with a locally transitive Cayley voltage space.*

And, vice versa, every locally transitive Cayley voltage space on a given graph H derives a regular covering projection associated with it.

Proof. If $p : G \to H$ is a covering, we can define the voltage group $A = CT(p)$ and the abstract fiber as the carrier set of A. From the definition of A we know that A fixes all fibers. And, as the covering is regular, $|A| = |p^{-1}u|$. Therefore we can label the vertices in each fiber by the elements of A and let A act on them by the left translation (this will work as the labeling l desired by the definition of association). Then we can set the voltages as right translation, so as to fulfill the condition $l(\overline{u} \circ W) = l(\overline{u}) \circ \xi_W$.

Now let (A, A, ξ) be a Cayley voltage space defined on a graph $G = (D, V, I, \lambda)$. We need to find a covering associated with this voltage space. First we need the covering graph - and that will be graph $G = (\overline{D}, \overline{V}, \overline{I}, \overline{\lambda})$ defined as follows:

- $\overline{D} = D \times A$
- $\overline{V} = V \times A$
- $\overline{I}(v, a) = (I(v), a)$
- $\overline{\lambda}(v, a) = (\lambda(v), a)$.

For this graph G, and $p : G \to H$ defined as the natural projection, p is a covering associated with the given voltage space. □

Chapter 4

Complexity of the covering problem

In this chapter we will show some of the results achieved in the area of computational complexity of graph covering. We suppose the reader knows the terms of polynomial time solvability and NP-completeness and won't extend about their meaning at this place. If there are any doubts, we refer to [8].

The problem can be formulated as follows:

The covering problem:	H-cover
Parameter:	A graph H.
Instance:	A graph G.
Question:	$\exists p$ covering, $p : G \to H$?

In other words: For a given graph H, take a graph G and decide, whether G does cover H, or not. In the following two sections we shall answer this problem for special classes of graphs.

4.1 Graphs for which H-cover is in P

As the title says, and as we have promised above, we will show here an example of a class of graphs for which the covering problem is solvable in polynomial time. The class of graphs is described by a simple property of its degree partition. Therefore let us first lay out the definition of the equitable partition of a graph and a term related to it - the degree matrix.

Definition 4.1.1. Let G be a graph. *Equitable partition* is a partition of the vertices of G into disjoint subsets $B_1, B_2 \ldots B_k$, called blocks, such that for each couple i, j holds that all vertices in B_i have the same number of neighbors in B_j.

The *minimal equitable partition* is minimal with respect to the number of the blocks.

Note that there always is at least one equitable partition, where each vertex forms its own block.

Definition 4.1.2. *Degree matrix* is a matrix related to an equitable partition of a graph, such that in the i-th row and j-th column is the number of neighbors of the vertices from B_i in block B_j. If we derive the matrix from the minimal equitable partition, we say we have to do with the *degree refinement matrix.*

Remark 4.1.3. The minimal equitable partition is defined uniquely, hence the degree refinement matrix is unique, too.

For better comprehension, look at the equitable partition in figure 4.1.

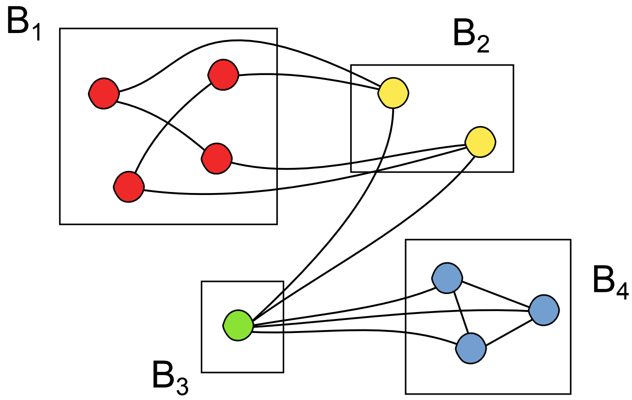

Figure 4.1: An equitable partition.

The degree matrix of this partition is:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

One might ask, given a graph G, how do I find its equitable partition? Algorithm 1 gives the answer. Its idea is that it begins with some "almost equitable partition" and then improves it slightly, until finally the real equitable partition is found. The outcome of this algorithm surely is an equitable partition. If it was not, then in some block there would be two vertices with a different number of edges going to an other block, which means they would have distinct vectors assigned by the algorithm, but then he algorithm should not have stopped yet. Short examination will show that this equitable partition really is the minimum one. Furthermore, it runs in time $O(|V_G|)^4$ and will be used in proof of theorem 4.1.4.

An illustration of the technique of this algorithm is in image 4.2, where there is a graph G and the first three steps of the algorithm: The first partition $B_{1,1} = V(G)$ and the labeling if the vertices with $d(u)$. Then according to this labeling the vertices of the graph are divided into three sets: the green one - $B_{2,1}$, the yellow one - $B_{2,2}$ and the blue one - $B_{2,3}$, and all vertices are assigned new vectors, which on the i-th position have the number of neighbors of the given vertex in the i-th part of the partition. Only those with the same vector remain in the same block after next redistribution. As in the last picture of image 4.2 all vertices in each block have the same vectors, the next partition would look exactly the same, which is the condition for the algorithm to stop and return this actual partition as the right one. We have found the equitable partition of graph G.

Algorithm 1 Minimum equitable partition.

Input: A graph G.
initialize: $\mathcal{B}_1 = B_{1,1} := \{V(G)\}$, $k_1 := 1$, $t := 1$
while $k_t \neq k_{t-1}$ **do**
  for $u \in V_G$ **do**
    for $i = 1 \ldots k_t$ **do**
      $d(u)_i := |N_u \cap B_{i,t}|$
    end for
  end for
  $t := t + 1$
  sort vectors $d(u)_i$ lexicographically
  for $i = 1 \ldots k_t$ **do**
    $B_{i,t} := \{u \in V(G), d(u)$ is the i-th vector in the sorted order$\}$
  end for
  $\mathcal{B}_t = \{B_{1,t}, \ldots B_{k_t,t}\}$
  $k_t := |\mathcal{B}_t|$ some of the vectors in $\mathcal{B}_t$
end while
return $\mathcal{B}_t$
Output: Minimal equitable partition $\mathcal{B}_t = \{B_{i,t}, i = 1, \ldots k_t\}$.

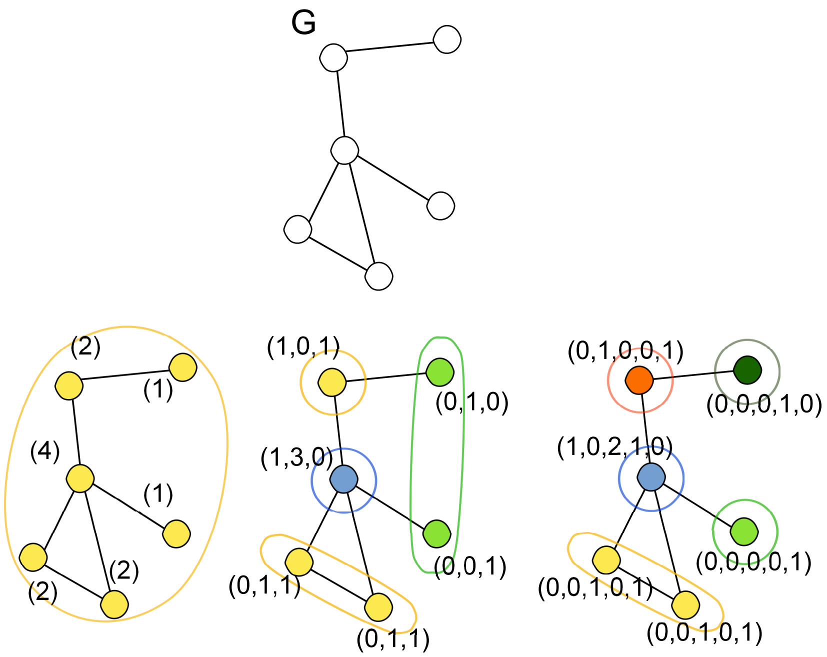

Figure 4.2: How algorithm 1 works.

After explaining these species, we can finally return to our previous object and show here an example of a class of graphs for which the covering problem is solvable in polynomial time.

Theorem 4.1.4. *If the only equitable partition of H is the partition into singletons, then the covering problem is in P.*

Proof. First we have to realize that if two vertices u and v of the covering graph are in the same block of the minimal equitable partition of G, then also their images $p(u)$ and $p(v)$ from H have to be in the same block of the minimal equitable partition of H. And it holds also reversely - all lifts of a vertex of one block of H have to be in the same block of G. As the only degree equitable partition of vertices of H is the one into singletons, the lifts of all vertices of H would be uniquely determined by the equitable partition of G. Using algorithm 1 we can find the minimum equitable partition in polynomial time. □

In the proof we used the fact that if G covers H, then two vertices $u, v \in G$ are in the same block of the minimal equitable partition of G if and only if their images $p(u), p(v) \in H$ are in the same block of the minimal equitable partition of H. I.e. they have the same degree refinement matrix. Interesting is that also the reverse implication holds, as says Leighton's theorem about common covers [16]. (Note that if G covers H, they share a common cover - G itself.)

Theorem 4.1.5. Leighton *For connected finite graphs H and $\bar{H}$, the following are equivalent:*

1. *H and $\bar{H}$ share a common finite cover,*
2. *H and $\bar{H}$ share a common (possibly infinite) cover,*
3. *H and $\bar{H}$ have the same degree refinement matrix.*

Kratochvíl, Pruskurowski and Telle [14] extended the result of theorem 4.1.4 also for graphs that have one, two or four vertices in each block of the partition. The proof uses a very nice reduction of this problem to a well known 2-SAT problem. We will prove here a constricted version of this theorem for graphs that have only one or two vertices in each partition.

Theorem 4.1.6. *If each block of the minimum equitable partition of H has at most two vertices, then the covering problem is in P.*

Proof. As we have already mentioned, we shall transform this problem into the 2-SAT problem (problem of satisfying a set of Boolean formulas with at most two literals), which is, according to [2], solvable in linear time.

We will begin by labeling the vertices of H which are in the "big" two-element blocks. One of them will be assigned

$$\text{TRUE},$$

and the other

$$\text{FALSE}.$$

Some of the vertices of G should be mapped by the covering projection onto vertices from these blocks - for every such vertex u we will create a Boolean variable x_u. Now, for each pair of vertices from the same block, joined by an edge, or having a common neighbor, we introduce clause

$$(x_u \vee x_v) \wedge (\neg x_u \vee \neg x_v).$$

Thereunto, if two blocks of H of size two are joined with an edge, then also the corresponding blocks of G are connected. And for each such edge (u, v) we introduce clause

$$(x_u \vee x_v) \wedge (\neg x_u \vee \neg x_v),$$

if the matching in H connects vertices which have the same value (both TRUE or both FALSE), and

$$(x_u \vee \neg x_v) \wedge (x_u \vee \neg x_v)$$

if they have different values (one is TRUE and the other FALSE).

Now we only have to think about what we have done. In fact, we have just rewritten the conditions for graph covering into some clauses. If all the clauses we generate can be satisfied, then the value assigned to variable x_u determines onto which of the two elements of the block should the vertex u be mapped by the covering projection - i.e. it unambiguously determines its image. $\square$

4.2 Graphs for which H-cover is NP-complete

In this section we will show a class of graphs for which the H-cover problem is NP-complete. We will begin with a not very well-known class of graphs, called solid graphs. Then we shall write a theorem saying that some k-regular graphs are solid [13] and so obtain a more concrete class of graphs for which the H-cover problem is hard. And, finally, we will show Fiala's method [7] to extend this result to all k-regular graphs.

First of all we have to set down the definition of partial covering, in order to be able to understand what a solid graph is.

Definition 4.2.1. Given two graphs G and H, mapping $\dot{p} : G \to H$ is said to be a *partial covering*, if it is locally injective, i.e. the neighborhood N_u of vertex u is injectively mapped onto the neighborhood of $N_{p(u)}$.

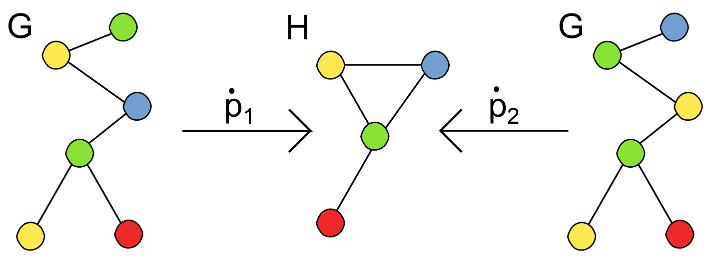

Figure 4.3: Graphs G ang H and partial covering $\dot{p}_1$ and mapping $\dot{p}_2$ that is *not* a partial covering.

Note that the difference between a covering and a partial covering is that for a covering we require the mapping to be locally *bijective* (see definition 2.0.27), while in the case of partial covering we draw in with *injectivity*. Thus each covering is also a partial covering, but, in general, it does not hold reversely. See also image 4.3, where there are two graphs G and H and two mappings

$\dot{p_1}, \dot{p_2} : G \to H$. The colors of the vertices indicate, which vertex they should be mapped on. Notice that $\dot{p_1}$ is a partial covering, but not a covering, while $\dot{p_2}$ is not even a partial covering.

Definition 4.2.2. Let H be a graph, u its vertex and $d = deg(u)$. Let us create graph H_u by splitting u into d leaves $u_1, \dots u_d$ - joining each of them to one of the neighbors of u, and leaving the rest of H as it was. H is called *solid*, if all partial covers $\dot{p_u} : H_u \to H$ can be after reunion of $u_1, \dots u_d$ into the original vertex u extended into an automorphism of H.

In image 4.4 there is a graph G and the "splitted" graph G_a. There are two options how one can partially cover G with G_a. In the case of $\dot{p_1}$ when $\dot{p_1}(a_1) = \dot{p_1}(a_2)$ we can define the automorphism $\alpha := \dot{p_1}$ for the vertices b, c, d, and define $\alpha(a) := \dot{p_1}(a_1) = \dot{p_1}(a_2)$. But in the other case there is no way of defining an automorphism that would be an extension of the partial covering $\dot{p_2}$, whereupon graph G is not solid.

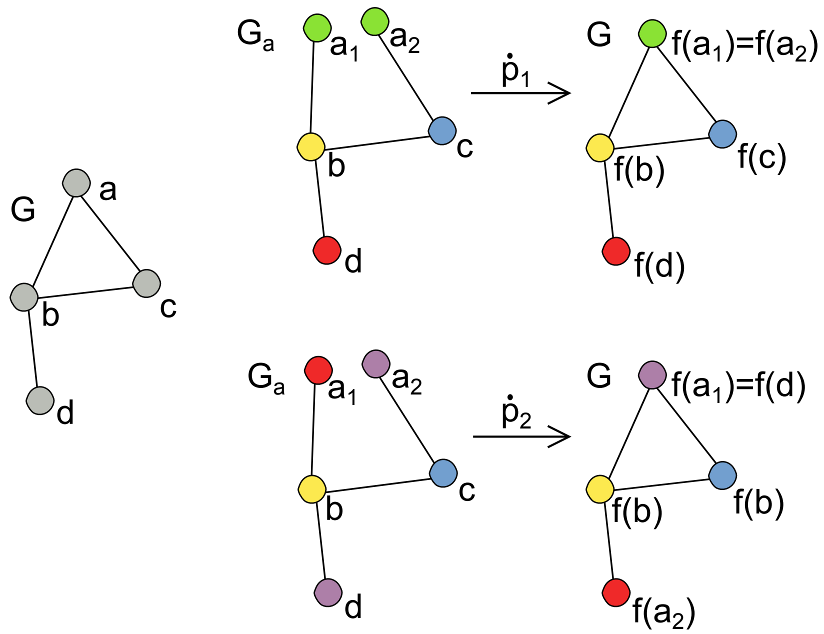

Figure 4.4: Graph G with its "splitted" graph G_a and its partial cover $\dot{p_1}$ and partial cover $\dot{p_2}$, which proves its non-solidity.

It was very easy to show that the graph in this example was not solid. It would be much more difficult to show that some graph actually *is* solid. Fortunately, a big class of solid graphs is described in the following theorem:

Theorem 4.2.3. *All k-edge colorable k-regular graphs and all* $\lceil \frac{k+2}{2} \rceil$*-edge-connected k-regular graphs are solid.*

This characterization is tight - there is an example of a $\lceil \frac{k+1}{2} \rceil$-edge-connected k-regular graph which is not solid [6].

Hence, an example of a solid graph can be also Q_3 in figure 2.5 or K_4 in figure 2.4 in the first chapter, as they fulfill both conditions of theorem 4.2.3.

And, the so expected theorems from [13] is:

Theorem 4.2.4. *For a solid graph H the H-cover problem is NP-complete.*

Theorem 4.2.5. *The H-cover is NP-complete for all k-regular graphs H, where $k \geq 3$.*

We will, finally, prove at least this theorem. In the proof we will us the Kronecker double cover, therefore let us first write what it is:

Definition 4.2.6. Given a graph G, a graph $\tilde{G} = G \times \mathbb{Z}_2$ such that its vertex set is

$$V(\tilde{G}) = V(G) \times \{0, 1\},$$

and the edge set is

$$E(\tilde{G}) = \{((u, 1), (v, 0)), ((u, 0), (v, 1)); (u, v) \in E(G)\},$$

is called a *Kronecker double cover of G*.

It is noticeable that Kronecker double cover actually is a covering graph of the original one.

Proof. (Of theorem 4.2.5.)

The proof will go as follows: first we will show that the problem is NP-complete for some graph $\tilde{H}$ and then we will show that the $\tilde{H}$-cover problem can be reduced to the H-cover problem.

We know from theorem 4.2.5 that the problem is NP-complete for all solid graphs H. Therefore we can suppose H is not solid. Then it also cannot be bipartite. If it was, according to König-Hall marriage theorem it would be k-edge colorable and (by theorem 4.2.4) solid.

Now we shall define graph $\tilde{H}$ as the Kronecker double cover of H. Then $\tilde{H}$ is k-edge colorable and k-regular, i.e. it fulfills the premises of theorem 4.2.4 and so is solid, hence according to theorem 4.2.5 the $\tilde{H}$-cover problem is NP-complete.

Hereafter, G covers $\tilde{H}$ if and only if G covers H and is bipartite.

The reverse implication is straightforward, as $\tilde{H}$ is bipartite, and hence each of its covers has to be bipartite, and $\tilde{H}$ covers H, and by the composition of the two coverings $p_1 : G \to \tilde{H}$ and $p_2 : \tilde{H} \to H$ we obtain a covering $p_1 \circ p_2 : G \to H$.

For the other implication we will assume G is bipartite and we have a covering $p : G \to H$. Let us take some fixed black and white coloring of the vertices in G. Now we will define a mapping $p_1 : G \to \tilde{H}$ so that for all white vertices of G will hold that $p_1(u) = (p(u), 1)$ and for the black ones $p_1(u) = (p(u), 0)$. We can easily see that p_1 is a covering, because the neighbors of a vertex $(u, 0)$ all have the second coordinate 1 and thanks p was a covering, we can be sure we did not omit any vertices. □

Chapter 5

Complexity of the regular covering problem

When one has seen the results in the previous chapter, natural question arises - how is it with regular covering? The condition on the covering to be regular is quite strong, but is it strong enough to make the problem polynomially solvable for all graphs H? We have not fully answered this question, but we have found that for some ratios $|G|/|H|$ it is in P, regardless of the structure of H or G. We will show our result in the second section of this chapter and finally we will be concerned with even more special type of coverings, where there will be some more claims on the covering transformation group (i.e. also on the voltage group).

5.1 p-regular cover problem

Before we inquire into the problem of H-regular cover deeper, we shall ask an easier question: Given graphs G, H and a covering projection p, how hard is it to decide, whether this covering is regular? In other words, we are asking about the computational complexity of the following problem:

The p-regular covering problem:		p-regular cover
Instance:	Graphs H and G.	
	A covering $p : G \to H$.	
Question:	Is p regular?	

The answer is to this question is very favorable:

Theorem 5.1.1. *For all graphs G and H, the p-regular cover problem is solvable in polynomial time.*

Proof. The proof will use the theory of voltage spaces. We know that each covering graph can be viewed as a voltage graph. In this proof we will find the appropriate voltage assignment of H that creates G and then we will try to find out whether this assignment creates a regular covering or not.

First we need s spanning tree T of G. When we have one, we have to divide G into $k := \frac{|G|}{|H|}$ copies of T. Then we assign all the edges in H from the spanning tree the identical permutation and according to the structure of G we assign the non-spanning tree edges the permutations on k elements (reversely to the technique described in remark 3.4.5): if the edge goes from the ith copy of T to the j-th and from the j-th to the k-th, the permutation assigned to this edge will look like this: $(\dots i, k, j, \dots)$.

Now when we have the voltage assignments we want to know what does the subgroup A generated by these permutations look like.

Recall that the Burnside's lemma (1.0.13) says that for a group A acting on the k copies of H holds:

$$|P| \sum_{O_x} 1 = \sum_{\pi \in P} |fix_\pi|.$$

(For the definition of O_x and fix_π see 1.0.12.)

As we are interested only in connected covering graphs, the action is to be transitive and there is only one orbit. Furthermore, if the covering is to be regular, then, according to theorem 3.4.6, the only element that fixes some points is the identity. Thereby the sum on the right side is only k, and $|A| = k$.

Hence we can successively generate all the elements of A and as soon as there are more than k we know that the covering is not regular. Otherwise we can check the permutations of A for fixed points. If there are none, the covering is regular, else it is not.

As we always have to generate at most $k+1$ permutations from A, the problem is solvable in polynomial time. □

5.2 H-regular cover problem

We shall now return to the question of computational complexity of the decision problem, whether there exists a regular covering for given graphs H and G. Let us bring in the problem formally:

The regular covering problem:	H-regular cover
Parameter:	A graph H.
Instance:	A graph G.
Question:	$\exists p$ regular covering, $p : G \to H$?

It shows up that the demand for regularity is so strong that the problem is polynomially solvable in many cases. And unlike the case of H-cover, the condition has nothing to do with the structure of G or H.

Theorem 5.2.1. *H-regular cover problem is polynomially solvable, if the ratio $|G|/|H|$ is a prime number.*

Proof. The principal idea of the proof is to look at the problem from the opposite side. For the given prime number $q = |G|/|H|$, we will construct all graphs (up to isomorphism) of size $q \times H$ that are regular covers of H and then check, if some of them is isomorphic with G. According to Luks [17], graph isomorphism can be, for bounded valence graphs, tested in polynomial time.

We are looking for covers of a given graph H, and from the definition of graph covering is clear that G has bounded maximal degree (in fact it is same as maximal degree of H). So there remains only the question how hard it is to create all regular covers of H of size $q \times H$, i.e. how many such graphs in fact do exist.

We will construct covers of H using permutation voltage assignments (section 3.4). What kind of permutations can be assigned to the edges? As mentioned in remark 3.4.5, the easiest way is to take a spanning tree T of H, and assign all the edges from T the identity element, and non-trivial elements assign only to the edges from $E(H) \setminus E(T)$. The main condition to make the corresponding covering regular (explicitly formulated in theorem 3.4.6), is that the group generated by the elements assigned to the edges cannot contain a permutation with a fixed point (except for the identity).

Now let us take q copies of H, $e_1, \dots e_k$ let be the edges from $E(H) \setminus E(T)$, and $\mu(e_1), \dots \mu(e_k)$ the permutations assigned to these edges, in the corresponding order. Then permutation group $P = \langle \mu(e_1), \dots \mu(e_k) \rangle$ generated by these elements acts on the copies of H. The condition that P cannot contain any element that would fix some point means in the terms of group actions that for each permutation from P the orbits of all elements are of the same size. If they were not, some edge would be assigned a permutation consisting of disjoint cycles of different size, let us denote these cycles σ_1 and σ_2. Let us denote their orders r_1 and r_2, respectively, and without lost of generality, let $r_1 < r_2$. Then involution of this permutation to the power r_1 is an element of P. However, instead of σ_1 there will be just identity, but $\sigma_2^{r_1}$ is not identity, thus P would contain an element that has a fixed point, but is not identity and that is a contradiction.

As we are acting on a set of size of a prime q, we get that the orbit can only be of size 1, or q. Therefore the only admissible groups P are cyclic groups, generated by one permutation of order q.

Hence have permutation groups, which contain only powers of one given (generating) permutation. Different choice of the generating element of the permutation group can be understood as different reordering of the q copies of the spanning tree at the beginning, hence we see that the two graphs constructed in this way would be isomorphic. Therefore we can consider only a permutation group given by a fixed generating permutation α. This group, as it is a cyclic group, is of order q.

There are q^k options of assigning the elements of the permutation group to the non-spanning tree edges, which is less than $q^{|H|} \leq |G|^{|H|}$, and that is, for a fixed graph H, just $|G|^r$, where r is constant, i.e. the problem is solvable in polynomial time. □

The procedure of building the catalog of all regular covers is described in algorithm 2. For given G and H we first choose some spanning tree - this can be done using any of the well-known algorithms, for example Kruskal's or Borůvka's. Then we set $q = |G|/|H|$ and $k = |E(H)| - |V(H)| + 1$, which is the number of the edges that are not in the spanning tree, and take a set $\mathcal{V}$ of all ordered combinations of k elements from the set $\{1, 2, \dots q\}$, possibly with repetition. Let us remark, that all these preparations can be done in polynomial time.

Remark 5.2.2. Note that given two graphs, whenever we are able to depict the groups of order of the graphs ratio, similar method to the one used in the previous case will work, which again yields from the connectedness of the sought graph and the theorem by Burnside:

$$|P| = |P| \sum_{O_x} 1 = \sum_{\pi \in P} |fix_\pi| = q_1 q_2.$$

Algorithm 2 Regular cover catalog.

Input: A graph H and prime q.
find a spanning tree T of H
$\bar{E} := E(H) \setminus E(T)$
set $k := |\bar{E}|$
create set $\mathcal{V} := \{f, f : \{1, 2, \dots q\} \rightarrow \{1, 2, \dots k\}\}$
for $m = 1, 2, \dots q$ **do**
  create a copy T_m of T
end for
for $f \in \mathcal{V}$, $j \in \{1, 2, \dots k\}$ **do**
  assign to the edges of H the $f(j) - th$ power of the cyclic permutation $\pi = (1, 2, \dots q)$
  for $e = u, v \in \bar{E}$ **do**
    create an edge from (u, i) to $(v, \pi^{f(j)}(e)(i))$ {create voltage graph G_j}
  end for
end for
Output: Set of graphs $\mathcal{G} = \{G_1, G_2, \dots G_{q^k}\}$.

The problem of finding regular covers then transforms to the problem of finding all regular permutation groups of a given order.

This appears to be quite easily done using some group theory for order of product of two primes q_1 and q_2. The first important fact reveals theorem 5.2.4. We shall not prove it here, one can find its proof in [12]. A special type of group - Frobenius group shall occur there, therefore let us show its definition:

Definition 5.2.3. *Frobenius group F_{q_1,q_2} is a group order $q_1 q_2$, where q_1, q_2 are primes and $q_2 | q_1 - 1$. It has two generators a and b that fulfill:*

- $a^{q_1} = b^{q_2} = 1$, and
- $b^{-1}ab = a^u$ for some $u \in \mathbb{Z}_{q_1}^*$ of order q_2.

Theorem 5.2.4. *Given a group A of order $q_1 q_2$, where q_1 and q_2 are primes, and $q_2 < q_1$, then either A is Abelian, or it is isomorphic to a Frobenius group F_{q_1,q_2}.*

Now it remains to detect, what kind of Abelian groups we have to consider. For this we will use the following lemma, which can be found for example in [12]:

Theorem 5.2.5. *Every finitely generated Abelian group is isomorphic to a direct sum of finitely many cyclic groups.*

Now we have everything necessary to prove that:

Theorem 5.2.6. *H-regular cover problem is polynomially solvable, if the ratio $|G|/|H|$ is equal to $q_1 q_2$ for q_1 and q_2 prime numbers ($q_1 \neq q_2$).*

Proof. The proof goes on in the same way as the proof of theorem 5.2.1. The question again is, elements of what group can be assigned to the non-spanning tree edges? According to the conclusion of remark 5.2.2, the groups has to be of order q_1q_2. Concluding from theorem 5.2.4, the group has to be Frobenius, or Abelian, and therefore (theorem 5.2.5) it is a direct sum of tow cycles of length q_1 and q_2, or it has just one generator and it is itself a cycle (of length q_1q_2).

Thus we have at most three finite groups to consider, and using the same arguments as in the proof of theorem 5.2.1, we can decide, whether G covers H in polynomial time. □

Note that in the case when $q_2 = 2$, the direct sum of cycles of length 2 and q_1 coincides with the dihedral group D_{2q_1} - the group of symmetries of a regular polygon with q_1 sides.

5.3 Special H-regular cover problems

According to theorem 3.4.9, there is up to isomorphism exactly one group assigned to a regular graph covering. Therefore we can strengthen our demands on the sought covering projection by laying some conditions on this group.

Useful condition turns out to be being Abelian. We then inquire about the complexity of the decision problem:

Abelian regular covering problem:		Abelian H-regular cover
Parameter:	A graph H.	
Instance:	A graph G.	
Question:	$\exists p$ regular covering, $p : G \to H$, with $CT(p)$ Abelian?	

Once more, we are asking about the complexity of the problem of covering a given graph H with graph G, by a covering projection whose covering transformation group (and also the voltage group) is Abelian.

This problem shows up not to be too difficult, thanks to the findings of group theory. First of all, observe:

Remark 5.3.1. There is theorem 5.2.5 saying that each finitely generated Abelian group is isomorphic to a direct sum of finitely many cyclic groups. When trying to follow the same routine as in the case of H-regular covering problem, generating all non-isomorphic relevant covers of H, the question of complexity of the Abelian H-regular covering problem transforms into the problem of finding all non-isomorphic Abelian groups of size $n := |G|/|H|$.

Therefore the first question would be: How many Abelian groups of given order are there? This is a well-studied problem and there are many estimates on the number. We shall use the one found in [22] and [20]. To cite it, let us first write down some notations.

Let $q_1^{k_1} q_2^{k_2} \dots q_j^{k_j}$ be a prime decomposition of n, $P(k_i)$ the number of unordered additive factorizations of number k_i, and finally let $a(n)$ be the number of Abelian groups of order n.

Then the following hold:

$$a(n) = \prod_{i=1}^{j} P(k_i),$$

$$P(k_i) \sim \frac{1}{4k_i\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}k_i}},$$

and for $k_i \geq k_i\sqrt{3}$

$$P(k_i) \sim e^{\pi\sqrt{\frac{2}{3}k_i}}.$$

One might object, that the number is exponential, but in fact, the sum of all exponents k_i of the prime decomposition is at most $\log(n)$. Hence we obtain

$$a(n) \leq \prod_{i=1}^{j} e^{\pi\sqrt{\frac{2}{3}k_i}} = \prod_{i=1}^{j} e^{\pi\sqrt{\frac{2}{3}k_i}} = e^{\pi\sqrt{\frac{2}{3}\sum_{i=1}^{j} k_i}} \leq e^{\pi\sqrt{\frac{2}{3}\log(n)}} \leq e^{\pi\sqrt{\frac{2}{3}}\log(n)} = n^{\pi\sqrt{\frac{2}{3}}},$$

thus the number of Abelian groups of order n is polynomial in n.

Let us now state the key theorem of this section:

Theorem 5.3.2. *Abelian H-regular covering problem is solvable in polynomial time.*

Proof. Using the observation of remark 5.3.1, we only need to find all Abelian groups of the given size. Thanks to the calculations above we already know there won't be too many of them, and at the end of this chapter we shall as well provide a routine which actually can generate all of them in polynomial time.

Having all Abelian groups, we can create corresponding voltage graphs of a given order using again algorithm 2 from the previous section, and finally check whether any of them is isomorphic to G, using [17]. If so, G covers H regularly, and the covering group is Abelian, as required.

The total complexity thus is the complexity of finding all Abelian groups multiplied by the complexity of algorithm 2 building the catalog of voltage graphs and algorithm checking isomorphism of the voltage graphs and graph G. As all these algorithms work in polynomial time, the result is polynomial, too. □

Although the problem of generating Abelian groups of a given order n is well studied, we decided to provide a straightforward algorithm to implement this task to make the proof self-complete.

From the fact, that finite Abelian groups are isomorphic to a direct sum of disjoint cycles, yields that to generate all Abelian groups of a given order we need to find all combinations of cycles of different length, whose direct sum will be of order n.

For example, if n is 12, there are 4 options:

$$C_2 \times C_2 \times C_3$$
$$C_4 \times C_3$$
$$C_2 \times C_6$$
$$C_{12}.$$

Noting that for distinct prime numbers $q_1, q_2, \ldots q_t$ holds that $C_{q_1} \times \ldots \times C_{q_t}$ is isomorphic to $C_{q_1 \ldots q_t}$ (they both have an element of order $q_1 \ldots q_t$), we can compose all Abelian groups using additive factorizations of the exponents of prime decomposition of the desired order of the group.

In our example, the prime decomposition is $12 = 2^2 \cdot 3$, and the decompositions of the exponents are:

Decompositions of 2	Decomposition of 1
2	1
$1+1$	

According to this, and the observations from previous page, there should be only $2 \cdot 1$ non-isomorphic Abelian groups of order 12. One can be written as $C_2 \times C_2 \times C_3$, and the other as $C_4 \times C_3$. And, in fact, for the other two groups listed above holds that $C_2 \times C_6 \simeq C_2 \times C_2 \times C_3$ and $C_{12} \simeq C_4 \times C_3$ (because $C_2 \times C_3 \simeq C_6$).

A short description of the procedure of generating Abelian groups would therefore be:

We first decompose n into prime factors and obtain $n = q_1^{k_1} q_2^{k_2} \ldots q_j^{k_j}$. Then we find all additive decompositions of the exponents.

Consequently, we take all possible j-tuples of these additive decompositions, and so create all direct sums of cycles of corresponding lengths.

The procedure of finding additive decomposition of a number is described in algorithm 3.

Before we initiate the algorithm, we assign the input integers values: $z := k_i$, $m := k_i, x := 1$, and $y := 1$, and set the row V^1 to be empty.

The algorithm works so, that it runs through all integers smaller or equal to the number decomposed and the number last used (so as not to duplicate the partitions), and in each step adds a number into a row. When a number is added, in case that the sum of numbers in a row is less than k_i, it continues by calling itself for the remainder. Otherwise it moves on to another row, filling in, if needed, the missing numbers of the decomposition from the previous rows.

The output will be the set $\mathcal{V}$ of all factorizations of k_i arranged "alphabetically" from the one using the largest numbers to the one consisting of k_i copies of number 1.

We should discuss the complexity of this algorithm. The highest number of steps is the length of the longest row, which is k_i, multiplied by the number of all rows, i.e. the number of all factorizations. That is, as it was already mentioned, $\frac{1}{4k_i\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}k_i}}$. We obtain, that the complexity is

$$log(n) \frac{1}{4 \log n\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}\log(n)}} = \frac{4}{\sqrt{3}} n^{\pi\sqrt{\frac{2}{3}}}.$$

This algorithm has to run for each exponent of the prime decomposition of n, hence it has to run at most $\log(n)$ times.

Algorithm 3 Additive decomposition.

Input: Integers z, m, x, y.
for $t = \min\{z, m\}, \dots 1$ **do**
  $V_y^x := t$
  if $m - t \neq 0$ **then**
    call Additive decomposition $\{m - t, t, x, y + 1\}$
  else
    s=1
    while $V_s^x = NULL$ **do**
      $V_s^x := V_s^{x-1}$
      $s++$
    end while
    add V^x to the list $\mathcal{V}$
    $x++$
  end if
end for
Output: Set of factorizations $\mathcal{V}$.

According to [21] we can factor integers in time

$$e^{c(\log(n))^{\frac{1}{3}}(\log\log(n))^{\frac{2}{3}}}$$

for some constant c depending on the implementation of the algorithm, and because $\log(n) \leq n$, it holds that

$$e^{c(\log(n))^{\frac{1}{3}}(\log\log(n))^{\frac{2}{3}}} \leq e^{c(\log(n))^{\frac{1}{3}}(\log)^{\frac{2}{3}}} = e^{c\log(n)} = n^c.$$

To obtain complete prime decomposition, this factorization has to be repeated at most $\log(n)$ times, therefore the prime decomposition will take time at most

$$\log(n)n^c.$$

It follows, that the total time needed for generating non-isomorphic Abelian groups of order n is

$$\frac{4}{\sqrt{3}}\log(n)n^{\pi\sqrt{\frac{2}{3}}+c}.$$

We are aware of the fact, that the algorithm provided is not the fastest possible, neither are our rounded estimates, but that was not our purpose. Our purpose was to find some algorithm, which would be fast enough to prove that the problem is polynomial, and in this we succeeded.

Conclusion

Throughout this work we have tried to gather fundamental knowledge on regular coverings of graphs and show all various approaches and the main results. In the second part we concentrated on the task of complexity of graph coverings, and tried to find some new results for the complexity of regular covering problem.

We were successful in this, proving that H-regular problem is polynomially solvable for all graphs G whose size is a prime multiple, or two-prime multiple of the size of the given graph H. However, the question still remains open for all other graphs, which will probably need a more sophisticated method, than our brute-force approach of generating all regular covers of the given size. Nevertheless, we have raised up another query - laying some demands on the nature of the group related to the covering - and our approach turned up to be helpful once more in this case. Thanks to it we concluded that for Abelian groups it can be solved in polynomial time.

An interesting task might be to try to adjust the graph isomorphism testing algorithm and construct the coverings straightforward, instead of constructing all regular covers and then just test if the graph is isomorphic to any of them. This, as well as extending the result achieved for Abelian groups also to solvable groups, which might be the aim of further research.

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List of Notations